

Return of the Superheroes

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1. Reduction

Let $B(n, p)$ be the required bottleneck value. Put

$$s = \frac{a}{2}, \quad c = b + s - s^2, \quad y = x + s. \quad (1)$$

Then, in $\mathbb{F}_p$,

$$T_{a,b}(x) + s = (x + s)^2 + c, \quad q_{c(y)} = y^2 + c, \quad y_0 = s. \quad (2)$$

Hence translation preserves the first collision and

$$f(a, b) = \mu_{c(s)} + \lambda_{c(s)}, \quad (3)$$

where μ and λ are the tail and cycle lengths in the functional graph of q_c . Grouping equal c and memoizing solved states evaluates every discovered (c, y) once.

For $t \in \mathbb{N}$, define the bipartite graph

$$G_t : (a, b) \in E(G_t) \quad \text{if and only if} \quad f(a, b) \geq t. \quad (4)$$

Then

$$B(n, p) = \max\{t : G_t \text{ has a perfect matching}\}. \quad (5)$$

Thus a perfect matching in G_t is a lower certificate; a Hall-deficient set in G_{t+1} is an upper certificate. In particular,

$$B(n, p) \leq \min\left(\min_{a \leq n} \max_{b \leq n} f(a, b), \min_{b \leq n} \max_{a \leq n} f(a, b)\right). \quad (6)$$

2. Required instance

For villain 493,

$$\max_{1 \leq a \leq 611} f(a, 493) = f(402, 493) = 349, \quad (7)$$

with unique maximizer $a = 402$. Since every pairing uses villain 493,

$$B(611, 14411) \leq 349. \quad (8)$$

The certificate contains a permutation $\pi \in S_{611}$ satisfying

$$f(402, \pi(402)) = f(402, 493) = 349, \quad \min_{a \neq 402} f(a, \pi(a)) \geq 350. \quad (9)$$

$$B(611, 14411) = 349. \quad (10)$$

□

3. Bonus

For $p = 17377$, the following four exact maxima suffice:

$$\max_{a \leq 490} f(a, 1) = 404, \quad \max_{b \leq 777} f(145, b) = 405, \quad (11)$$

$$\max_{b \leq 999} f(759, b) = 408, \quad \max_{a \leq 923} f(a, 513) = 405. \quad (12)$$

At threshold 409, an isolated vertex is given by

$$\begin{cases} \text{villain 1, } 2 \leq n \leq 490 \\ \text{hero 145, } 491 \leq n \leq 777 \\ \text{hero 759, } 778 \leq n \leq 999. \end{cases} \quad (13)$$

Therefore $B(n, 17377) \leq 408$ for every $2 \leq n \leq 999$.

At threshold 408, prefixes below 924 are excluded by

$$\begin{cases} \text{villain 1,} & 2 \leq n \leq 490 \\ \text{hero 145,} & 491 \leq n \leq 777 \\ \text{villain 513,} & 778 \leq n \leq 923. \end{cases} \quad (14)$$

A certified $\pi_{924} \in S_{924}$ satisfies

$$\min_{a \leq 924} f(a, \pi_{924}(a)) = 408. \quad (15)$$

Short alternating-path updates extend this matching through every $n = 925, \dots, 999$. Hence

$$\max_{2 \leq n \leq 999} B(n, 17377) = 408, \quad \arg \max_{2 \leq n \leq 999} B(n, 17377) = \{924, 925, \dots, 999\}. \quad (16)$$

□